CONDITIONS FOR A MINIMUM ENTROPY-PRODUCTION RATE IN A HEAT-CONDUCTING BODY

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Let us consider the nonequilibrium states of an isotropic solid body at whose boundaries steady temperatures are maintained. When the only irreversible process is heat transfer by conduction at λ = const and the thermal expansion of the body is negligible, the generalized force X, the thermal flux I, the phenomenological coefficient L, and the entropy-production rate dS_i/dt are determined by the following relations of nonequilibrium thermodynamics [1]:

$$X = \operatorname{grad}\frac{1}{T}, \tag{1}$$

$$I = LX, \tag{2}$$

$$L = \lambda T^2, \tag{3}$$

$$\frac{dS_i}{dt} = \int_V IX dV = \lambda \int_V T^2 \operatorname{grad}^2 \frac{1}{T} dV. \tag{4}$$

If we introduce the new variable

$$\tau = \ln T, \tag{5}$$

then expression (4) can be given a form more convenient for analysis:

$$\frac{dS_i}{dt} = \lambda \int_V \operatorname{grad}^2 \tau \, dV. \tag{6}$$

Under steady boundary conditions, the field of τ that corresponds to the minimum entropy-production rate must satisfy the Euler-Ostrogradskii variational equation

$$\operatorname{div}\operatorname{grad}\tau = 0 \tag{7}$$

or, in Cartesian coordinates x, y, z,

$$\frac{\partial^2 \tau}{\partial x^2} + \frac{\partial^2 \tau}{\partial y^2} + \frac{\partial^2 \tau}{\partial z^2} = 0. \tag{8}$$

Thus, at minimum dS_i/dt, the field of τ is described by a Laplace equation.

Substitution of (5) into (8) gives an equation for the temperature field T (x, y, z):

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{T}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right]. \tag{9}$$

Under steady boundary conditions, Eq. (9) determines the steady temperature field. Since

$$\operatorname{div} I = -\lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right),$$

it follows from (9) that to obtain a temperature field T(x, y, z) that satisfies the necessary conditions for minimum dS_i/dt, the body must have heat sinks (div I < < 0) with the intensity

$$\operatorname{div} I = -\lambda \cdot \frac{1}{T}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right]. \tag{10}$$

When the entropy-production rate is minimal, the energy losses associated with irreversible heat transfer in the body

$$E = T_0 \frac{dS_i}{dt}$$

are also minimal. Therefore, the introduction of heat conductors connected to heating or cooling machines into the heat insulation is one method of reducing energy losses [2, 3]. The maximum efficiency of such systems is estimated by calculating dS_i/dt and E for the field of τ in the insulating structure that satisfies Eq. (8). Naturally, all of the conventional analytic and numerical methods of solving the Dirichlet problem and the methods of modeling potential fields can be used in solving Eq. (8).

Equation (9) and its equivalent (10) admit simple physical interpretation. Let us consider the steady filtration through a heat-conducting body of a liquid (gas) with a constant volume specific heat and a negligible (as compared with λ) thermal conductivity. We shall assume that there is reversible heat transfer between the fluid particles and the body.

When all of these conditions are satisfied, fluid filtration produces in the body thermal fluxes I' for which

$$\operatorname{div} I' = -c\bar{w}\operatorname{grad} T. \tag{11}$$

Considering the fact that

$$\frac{1}{T}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right] = \operatorname{grad} T \operatorname{grad}\tau,$$

relation (10) can be written as

$$\operatorname{div} I = -\lambda \operatorname{grad} T \operatorname{grad} \tau.$$

Therefore, when

$$\bar{w} = \frac{\lambda}{c}\operatorname{grad}\tau, \tag{12}$$

then

$$\operatorname{div} I = \operatorname{div} I'.$$

Since in this case the velocity field $\bar{w}$ is a potential field, condition (12) can be satisfied. The temperature field satisfies Eq. (9) and the entropy-production rate is minimal.

NOTATION

T is the absolute temperature; τ is the natural logarithm of T; S_i is the entropy production; λ is the thermal conductivity; X is the generalized force; I is the heat flux; L is the phenomenological coefficient; t is the time; V is the volume; x, y, and z are Cartesian coordinates of points on the body; T_0 is the ambient temperature; E is the exergic loss; c is the volume heat capacity; $\vec{w}$ is the filtration-rate vector of liquid (gas).

REFERENCES

1. S. de Groot and P. Mazur, Nonequilibrium Thermodynamics [Russian translation], Izd. Mir, 1964.

2. V. S. Martynovskii and I. M. Shnaid, Teplofizika vysokikh temperatur, no. 5, 1964.

3. G. A. Vikhorev, L. Z. Mel'tser, V. T. Cheilyakh, and I. M. Shnaid, Kholodil'naya tekhnika, no. 1, 1966.

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